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Non-Integrability Criteria for Hamiltonians in the Case of Lamé Normal Variational Equations

JUAN J. MORALES-RUIZ

Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Pau Gargallo 5, 08028 Barcelona, Spain

AND

CARLES SIMÓ

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We consider complex analytic classical Hamiltonian systems with two degrees of freedom and an invariant plane. Furthermore we assume that the normal variational equations (NVE) are of Lamé type. We describe the possible potentials giving rise to this kind of problems and non-integrability criteria based on the differential Galois approach to the Ziglin theorem. Some examples are included.

1. INTRODUCTION

A very important question concerning both theoretical and practical studies on Hamiltonian systems is integrability. By integrability we understand the complex analytic Liouville–Arnold integrability, that is, the existence of n functional independent complex analytic first integrals in involution, where n is the number of degrees of freedom of the Hamiltonian [1, 2]. Ziglin [23] derived necessary conditions for the existence of n such first integrals in a vicinity of a given solution, even without the involution assumption. His result was reinterpreted in terms of differential Galois theory in [3, 5, 16]. The basic idea, in our point of view, is that the integrability of the original system implies the solvability of the NVE in the differential Galois sense [15]. That means that the general solution of the NVE is obtained, from the coefficients field, by algebraic functions, quadratures and exponentiation of quadratures [11, 12, 13, 20].

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Very simple examples in two degrees of freedom lead to NVE [24, 10] of Lamé type [9, 21]:

$$\frac{d^2\xi}{dt^2} = (A\wp(t) + B)\,\xi, \tag{1}$$

where $\wp$ denotes the Weierstraß function and A and B are, in general, complex parameters. We recall that $\wp$ is a solution of the differential equation [21]

$$\dot{z}^2 = C(z), \tag{2}$$

where $C(z)$ is a cubic polynomial which can be reduced to the form

$$C(z) = 4z^3 - g_2 z - g_3. \tag{3}$$

It is assumed, in what follows, that the roots of C are simple (otherwise $\wp$ reduces to simpler functions). This is ensured if the discriminant of (3)

$$\Delta := 27g_3^2 - g_2^3 \tag{4}$$

is non-zero.

Hence (1) depends, in fact, on four parameters: A, B and the two coefficients of (3), g_2, g_3, known as invariants of $\wp$. If we set $A = n(n+1)$, it was observed, in the early examples [10], that the systems satisfying the necessary condition for integrability had n integer. The motivation of this paper was to understand this behaviour and to complete it.

In the forthcoming sections we obtain, first, the potentials of classical Hamiltonians with an invariant plane such that the NVE are of Lamé type. Then non-integrability criteria are obtained for these Hamiltonians. The results given here are not complete because we have not been able to prove that some numerical coefficients are different from zero (regardless this has been checked for a big number of them!). We conjecture that all of them are different from zero.

The paper ends with the study of some old and new examples. The case of the homogeneous Hénon–Heiles potential is studied in detail, including the dynamical meaning of the non-integrabiblity.

2. Computation of the Potentials

Let

$$H = \tfrac{1}{2}(y_1^2 + y_2^2) + V(x_1, x_2)$$

be a two degrees of freedom classical Hamiltonian, where V is a real analytic function on some domain which will be considered in $\mathbb{C}^2$. Assume that it exists a continuous family of integral curves, Γ_h, parametrized by the energy, h, lying on an invariant plane that, for concreteness, will be taken as

$$\Gamma_h\colon x_2 = y_2 = 0, \quad x_1 = x_1(t, h), \quad y_1 = y_1(t, h).$$

A necessary and sufficient condition is that

$$V(x_1, x_2) = \varphi(x_1) - \tfrac{1}{2}\alpha(x_1)\, x_2^2 + O(x_2^3), \tag{5}$$

where φ and α are arbitrary functions.

The NVE along Γ_h are [10]

$$\ddot{\xi} - \alpha(t, h)\, \xi = 0, \tag{6}$$

where, for simplicity, we denote by $\alpha(t, h)$ which, in fact, is $\alpha(x_1(t, h))$, and $x_1(t, h)$ is a solution of

$$\tfrac{1}{2}\dot{x}_1^2 + \varphi(x_1) = h,$$

the energy h ranging in a real interval.

We want to obtain a potential, V, of the type (5) (that is, to obtain the functions φ and α, the $O(x_2^3)$ being arbitrary) such that (6) is of the type (1), that is

$$\alpha(t, h) = A(h)\, \wp(t, h) + B(h), \tag{7}$$

A and B being parameters and $\wp$ the Weierstraß elliptic function. From now on we keep in mind that everything depends on h, but we do not write it explicitely.

From (6) and using $'$ to denote d/dx_1, it follows that

$$\dot{\alpha}^2(t) = 2\, \alpha'^2(x_1)\, h - 2\, \alpha'^2(x_1)\, \varphi(x_1). \tag{8}$$

Assume that $\alpha(x_1)$ is not identically constant. Hence, we can obtain $x_1 = x_1(\alpha)$ (possibly multivaluated). Hence, as from (7), (2), and (3) it follows that $\dot{\alpha}^2$ is a cubic polynomial in α, also depending on h, by comparing with (8) we get

$$\dot{\alpha}^2 = P(\alpha, h) = P_1(\alpha) + hP_2(\alpha), \tag{9}$$

where P is a polynomial of degree 3 in α and, therefore, either P_1 or P_2 must have degree 3.

Remark 1. The case $\alpha =$ constant $= B$ gives a separable potential up to the $O(x_2^3)$ terms. This is equivalent to $P_2 \equiv 0$ (see (10) below).

Hence, by comparing (8) and (9) and denoting by $\varphi(\alpha)$ the function $\varphi(x_1(\alpha))$ we have

$$\varphi(\alpha) = -\frac{P_1(\alpha)}{P_2(\alpha)}, \qquad \alpha'^2(x_1) = \tfrac{1}{2} P_2(\alpha(x_1)). \tag{10}$$

From (10) we obtain the potential from P_1 and P_2 by using the scheme

$$\begin{tikzcd}
P_2(\alpha) \arrow[r] \arrow[dr] & \alpha(x_1) \arrow[r] & \varphi(x_1) \\
P_1(\alpha) \arrow[r] & \varphi(\alpha) \arrow[ur] &
\end{tikzcd}$$

Let g_2 and g_3 be the invariants of $\mathfrak{P}$. Now we look for a relation between h, P_1, and P_2, from one side, and A, B, g_2, and g_3, from the other. From (7) we obtain

$$u := \mathfrak{P}(t) = \frac{1}{A}(\alpha(t) - B), \qquad v := \dot{\mathfrak{P}}(t) = \frac{\dot{\alpha}(t)}{A},$$

$$\dot{\alpha}^2(t) = A^2 v^2 = A^2(4u^3 - g_2 u - g_3).$$

Therefore

$$P(\alpha, h) = \dot{\alpha}^2(t) = \frac{4}{A}\alpha^3 - \frac{12B}{A}\alpha^2 + \left(\frac{12B^2}{A} - g_2 A\right)\alpha - \frac{4B^3}{A} + g_2 AB - g_3 A^2. \tag{11}$$

We introduce the coefficients $a_1, ..., d_2$ by setting

$$P(\alpha, h) = (a_1 + h a_2)\,\alpha^3 + (b_1 + h b_2)\,\alpha^2 + (c_1 + h c_2)\,\alpha + d_1 + h d_2. \tag{12}$$

By comparing (11) and (12) we obtain

$$\begin{aligned}
\frac{4}{A} &= a_1 + h a_2, \\
\frac{12B}{A} &= b_1 + h b_2, \\
-\frac{12B^2}{A} - g_2 A &= c_1 + h c_2, \\
-\frac{4B^3}{A} + g_2 AB - g_3 A^2 &= d_1 + h d_2.
\end{aligned} \tag{13}$$

Let us proceed to the effective computation of the potentials. We classify them according to the degree of P_2 and then we use (10). In the expressions below e denotes an integration constant. We shall restrict ourselves to real solutions.

(A) $\deg P_2=0 \Rightarrow P_2(\alpha)=d_2>0$, $\alpha=\pm\sqrt{d_2/2}\,x_1+e$.

(B) $\deg P_2=1 \Rightarrow P_2(\alpha)=c_2\alpha+d_2$, $\alpha=(c_2/8)\,x_1^2+ex_1+(2e^2-d_2)/c_2$.

(C) $\deg P_2=2 \Rightarrow P_2(\alpha)=b_2\alpha^2+c_2\alpha+d_2$. Let $D:=c_2^2-4b_2d_2$. We should consider three cases:

(C.1) $D=0$, $b_2>0$: $P_2(\alpha)=b_2(\alpha+c_2/(2b_2))^2$, $\alpha=e\cdot\exp((b_2/2)^{1/2}\,x_1)-c_2/(2b_2)$.

(C.2) $D<0$, $b_2>0$: $\alpha=\sqrt{-D}/(2b_2)\sinh((b_2/2)^{1/2}\,x_1+e)-c_2/(2b_2)$.

(C.3.1) $D>0$, $b_2>0$, $\alpha=\sqrt{D}/(2b_2)\cosh((b_2/2)^{1/2}\,x_1+e)-c_2/(2b_2)$.

(C.3.2) $D>0$, $b_2<0$, $\alpha=\sqrt{D}/(2b_2)\sin((-b_2/2)^{1/2}\,x_1+e)-c_2/(2b_2)$.

(D) $\deg P_2=3$. From (10) one derives

$$\alpha(x_1)=C\beta(x_1)+E, \qquad \text{with} \quad \beta'^2(x_1)=4\beta^3-\bar{g}_2\beta-\bar{g}_3,$$

where

$$a_2=\frac{8}{C},\ b_2=-\frac{24E}{C},\ c_2=-2C\bar{g}_2+\frac{24E^2}{C},\ d_2=-2C^2\bar{g}_3+2CE\bar{g}_2-8\frac{E^3}{C}.$$

Let $\bar{\Delta}=27\bar{g}_3^2-\bar{g}_2^3$. There are three possibilities:

(D.1) $\bar{\Delta}\neq 0$, and then $\beta=\wp(x_1+e)$.

(D.2) $\bar{\Delta}=0$ and two of the roots of $4\beta^3-\bar{g}_2\beta-\bar{g}_3$ are equal. There are two subcases (see, for instance, [9, Vol. I p. 27]). Let $\bar{e}_3\leqslant\bar{e}_2\leqslant\bar{e}_1$ the roots.

(D.2.1) $\bar{e}_2=\bar{e}_3=-\frac{1}{2}\bar{e}_1<\bar{e}_1$. Then

$$\beta=-\frac{3}{2}\frac{\bar{g}_3}{\bar{g}_2}+\frac{9}{2}\frac{\bar{g}_3}{\bar{g}_2}\operatorname{cosec}^2\left(\left(\frac{9\bar{g}_3}{2\bar{g}_2}\right)^{1/2}x_1+e\right).$$

(D.2.2) $\bar{e}_1=\bar{e}_2=-\frac{1}{2}\bar{e}_3>\bar{e}_3$. Then

$$\beta=\frac{3}{2}\frac{\bar{g}_3}{\bar{g}_2}-\frac{9}{2}\frac{\bar{g}_3}{\bar{g}_2}\coth^2\left(\left(-\frac{9\bar{g}_3}{2\bar{g}_2}\right)^{1/2}x_1+e\right).$$

(D.3) $\bar{\Delta}=0$ and the three roots are equal. We can write $P_2(\alpha)=a_2(\alpha-\bar{e}_1)^3$ and, hence,

$$\alpha=\frac{8}{a_2(x_1+e)^2}+\bar{e}_1.$$

In all the cases, when α is available as a function of x_1, the function φ is obtained from (10). We refer to Section 5, where some realizations of the above cases are given explicitely.

3. DIFFERENTIAL GALOIS SOLVABILITY FOR THE LAMÉ EQUATION

Given a system of linear differential equations, the study of the solvability in finite terms can be made by using the linear differential Galois theory (Picard–Vessiot theory). The coefficients are considered in a suitable differential field and the solutions in a differential extension of this field (the Picard–Vessiot extension), in an analogous way to the classical Galois theory. From now on by solvability of a linear equation we shall understand solvability in the differential Galois sense (see Introduction).

The known cases of solvability for the Lamé equation (1) are the mutually excluent following ones:

(i) The Lamé and Hermite solutions [9, 21, 17, 6]. In this case $n \in \mathbb{N}$ and the other parameters are arbitrary.

(ii) The Brioschi–Halphen–Crawford solutions [9, 17, 4, 6]. Now $m := n + \frac{1}{2} \in \mathbb{N}$ and B, g_2, g_3 satisfy an algebraic equation

$$0 = Q_m\left(\frac{g_2}{4}, \frac{g_3}{4}, B\right) \in \mathbb{Z}\left[\frac{g_2}{4}, \frac{g_3}{4}\right][B],$$

where Q_m has degree m in B. This polynomial is known as Brioschi determinant and will be discussed later on.

(iii) The Baldassarri solutions [4]. The condition on n is $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$, with additional conditions on g_2, g_3 and B.

Now we assume that the Lamé equation appears as NVE of a Hamiltonian with two degrees of freedom. In [16] it is proved that, in this case, there are no more cases of solvability of the Lamé equation than the preceeding ones.

For the application in Section 4, we shall analyze now some aspects of cases (ii) and (iii). We recall that the moduli on an elliptic curve $v^2 = 4u^2 - g_2 u - g_3$ is determined by the value of the modular function j:

$$j = j(g_2, g_3) = \frac{g_2^3}{g_2^3 - 27 g_3^2}. \tag{14}$$

That is, two elliptic curves are birationally equivalent if and only if they have the same value of the modular function (see, for instance [19]).

Despite the additional conditions on g_2, g_3, and B in the case (iii) are difficult to systematize, there is, in that case, the following result of Dwork answering a question posed by Baldassarri in [4].

LEMMA 1 [7]. *In the case* (iii) *above for a fixed value of n, the number of couples* (j, B) *is finite.*

For the families of type (ii) we recall the computation of the Brioschi determinant following Baldassarri [4]. If in the Lamé equation we make the Halphen substitution [9] $t=2z$ and we use the addition theorem for $\mathfrak{P}$ (see [21]) we obtain

$$\frac{d^2\xi}{dz^2}-4\left[n(n+1)\left(\frac{1}{4}\left(\frac{\mathfrak{P}''(z)}{\mathfrak{P}'(z)}\right)^2-2\mathfrak{P}(z)\right)+B\right]\xi=0. \tag{15}$$

If $(2\omega_1, 2\omega_3)$ are the periods of $\mathfrak{P}$, the singularity of (1) at $t=0$ (modulo the periods) is transformed to the singularities of (15) : $z=0, \omega_1, \omega_2, \omega_3$ (modulo the periods), where $\omega_1+\omega_2+\omega_3=0$. Now, to complete the Halphen transformation, we perform the change $\xi=(\mathfrak{P}'(z))^{-n}\,w$, obtaining

$$\frac{d^2 w}{dz^2}-2n\,\frac{\mathfrak{P}''(z)}{\mathfrak{P}'(z)}\frac{dw}{dz}+4(n(2n-1)\,\mathfrak{P}(z)-B)\,w=0,$$

with singularities as above. Let now $x=\mathfrak{P}(z)$ be the new independent variable. We have

$$\left(-x^3+\frac{g_2}{4}x+\frac{g_3}{4}\right)\frac{d^2 w}{dx^2}+\left(3x^2-\frac{g_2}{4}\right)(m-1)\frac{dw}{dx}$$
$$+[B-(2m-1)(m-1)\,x]\,w=0, \tag{16}$$

having singularities at ∞, e_1, e_2, e_3 (corresponding to the previous ones 0, ω_1, ω_2, ω_3 in (15)). We recall that $m=n+\frac{1}{2}$.

The exponents associated to the singularities are $(0, m)$ at e_i, $i=1, 2, 3$, and $(-2m+1,\ -m+1)$ at ∞. As the difference is $m\in\mathbb{N}$ there will appear, in general, logarithmic terms. But if in one of the singularities there are not logarithmic terms they do not appear in any of the other singularities, because all the singularities come from the unique singularity of (1) by means of the Halphen transformation. Furthermore, if in an equation over $\mathbb{P}^1$ all the exponents are integers and there are no logarithmic terms, then the general solution is rational. In particular, if this happens in (16), then we have solvability for the Lamé equation.

To avoid logarithmic terms at $x=\infty$, a necessary and sufficient condition is the existence of a Laurent series solution of the form

$$w=\sum_{j=0}^{\infty} c_j\,x^{2m-j-1}, \qquad c_0\neq 0 \tag{17}$$

corresponding to the lower exponent $-2m+1$.

This leads to a recurrent system for the coefficients $c_0, c_1, ...,$ which, in particular, gives the uncoupled system:

$$\begin{aligned}
&Bc_0 + (m-1)\,c_1 = 0,\\
&(2m-1)(m-1)\frac{g_2}{4}c_0 + Bc_1 + 2(m-2)\,c_2 = 0,\\
&(2m-1)(2m-2)\frac{g_3}{4}c_0 + (2m-2)(m-2)\frac{g_2}{4}c_1 + Bc_2 + 3(m-3)\,c_3 = 0,\\
&(2m-2)(2m-3)\frac{g_3}{4}c_1 + (2m-3)(m-3)\frac{g_2}{4}c_2 + Bc_3 + 4(m-4)\,c_4 = 0,\\
&\vdots\\
&(m+3)(m+2)\frac{g_3}{4}c_{m-4} + (m+2)2\frac{g_2}{4}c_{m-3} + Bc_{m-2} + (m-1)\,1c_{m-1} = 0,\\
&(m+2)(m+1)\frac{g_3}{4}c_{m-3} + (m+1)\,1\frac{g_2}{4}c_{m-2} + Bc_{m-1} = 0.
\end{aligned}$$

Therefore, the necessary and sufficient condition to have a solution of the form (17) is

$$0 = Q_m\left(\frac{g_2}{4}, \frac{g_3}{4}, B\right)$$

$$:= \begin{vmatrix}
B & m-1 & & & & \\
(2m-1)(m-1)\frac{g_2}{4} & B & 2(m-2) & & & \\
(2m-1)(2m-2)\frac{g_3}{4} & (2m-2)(m-2)\frac{g_2}{4} & B & 3(m-3) & & \\
 & (2m-2)(2m-3)\frac{g_3}{4} & (2m-3)(m-3)\frac{g_2}{4} & B & 4(m-4) & \\
 & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & m-1 \\
 & & & (m+2)(m+1)\frac{g_3}{4} & (m+1)\frac{g_2}{4} & B
\end{vmatrix}, \tag{18}$$

where all the non displayed entries of the determinant are zero.

4. Non-integrability Criteria

Given a complex analytic Hamiltonian system of two degrees of freedom having an integral curve, Γ, in [16] the following result is proved.

Theorem 1. *If the NVE along Γ is of Lamé type and falls outside cases* (i), (ii), (iii) *of Section* 3, *then the Hamiltonian system has no meromorphic first integrals independent of the Hamiltonian in a neighbourhood of Γ.*

When there is not any global meromorphic first integral independent of the Hamiltonian, we say that the system is not integrable.

Next we shall study the non-integrable cases of the families of potentials (A), (B), (C), and (D) obtained in Section 2, by using Theorem 1. We notice that, for a given potential (that is, given the polynomials P_1 and P_2 of Section 2) we have a one parameter family of Lamé equations, the parameter being the level of energy, h.

The analysis of the (D) families, which have $a_2 \neq 0$, is elementary. It is enough to consider that when h changes, according to (13) we reach irrational values of n. Hence, all the (D) families are non integrable.

Therefore we can assume $a_2 = 0$ (families (A), (B), (C)) so that for a given potential the value of n remains fixed when h changes. In particular one can not jump from one of the cases (i), (ii), (iii) to another one when h changes.

If our system falls in case (i) we can not derive additional integrability conditions from the analysis of the NVE. Before proceeding to the case (ii) we analyze the case (iii). Hence, assume that for a given value of h we have a Lamé equation of type (iii). We have $a_1 = 4/(n(n+1))$, with n as stated in Section 3 (iii).

Lemma 2. *Consider the curve $\sigma: h \to (j(h), B(h))$ defined by means of* (13) *and* (14) *with $a_2 = 0$. Then σ changes continuously with respect to h except in the cases*:

(1) $P_2 \equiv 0$, *which by Remark* 1 *has $n = 0$ and, hence, it is not really of Lamé type*,

(2) $b_2 = 0,\ c_2 = 0,\ b_1^3 - 3c_1 a_1 = 0$,

(3) $b_2 = 0,\ c_2 b_1 - 3a_1 d_2 = 0,\ 2b_1^3 - 9a_1 b_1 c_1 + 27a_1^2 d_1 = 0$.

Proof. From (13) we should have $b_2 = 0$ (otherwise B changes linearly with h). The possibilities to have $j(h)$ constant are $g_2(h) \equiv 0$, $g_3(h) \equiv 0$ (both cases can not occur simultaneously because Δ, as defined by (4), must satisfy $\Delta \not\equiv 0$) or both g_2 and g_3 independent of h.

The condition $g_2 \equiv 0$ gives (2) and $g_3 \equiv 0$ gives (3). If g_2 and g_3 do not depend on h one finds $c_2 = d_2 = 0$. But as $a_2 = b_2 = 0$ we have $P_2 \equiv 0$. ∎

From Lemmas 1 and 2 and Theorem 1 it follows immediately the next result.

PROPOSITION 1. *If* $a_2=0$, *the NVE are of Lamé type, not of the types* (i) *or* (ii) *and the Hamiltonian system is integrable, then one should have* $b_2=0$ *and either* $c_2=0$, $b_1^3-3c_1a_1=0$ *or* $c_2b_1-3a_1d_2=0$, $2b_1^3-9a_1b_1c_1+27a_1^2d_1=0$.

Now we start the discussion of case (ii).

To have integrability a necessary condition is $Q_m(g_2/4, g_3/4, B)\equiv 0$ as a polynomial in h, provided $\Delta\neq 0$. We want to express the conditions for integrability in terms of the coefficients $a_i, b_i, c_i, d_i, i=1,2$ of the polynomials appearing in the potential. We recall that in the case (ii) one should have $a_1=16/(4m^2-1)$ for some $m\in\mathbb{N}$, $a_2=0$, the remaining coefficients being arbitrary, except that b_2, c_2 and d_2 can not be zero simultaneously.

Let us introduce $\bar{B}=\bar{b}_1+h\bar{b}_2\equiv b_1+hb_2$, $\bar{C}=\bar{c}_1+h\bar{c}_2\equiv(c_1+hc_2)(a_1/16)$, $\bar{D}=\bar{d}_1+h\bar{d}_2\equiv(d_1+hd_2)(a_1^2/64)$. Then $B=\bar{B}/(3a_1)$, $g_2/4=\bar{B}^2/48-\bar{C}$, $g_3/4=-\bar{B}^3/864+\bar{B}\bar{C}/12-\bar{D}$ and the discriminant has the expression

$$\Delta=\bar{B}^3\bar{D}-\bar{B}^2\bar{C}^2-72\bar{B}\bar{C}\bar{D}+64\bar{C}^3+432\bar{D}^2.$$

Multiplying each column in (18) by 864 we get the $m\times m$ determinant

$$\mathcal{D}_m(\bar{B},\bar{C},\bar{D})$$

$$:=\begin{vmatrix} 18(4m^2-1)\bar{B} & 864(m-1) & & & & \\ (2m-1)(m-1)W_1 & 18(4m^2-1)\bar{B} & 864\cdot 2(m-2) & & & \\ (2m-1)(2m-2)W_2 & (2m-2)(m-2)W_1 & 18(4m^2-1)\bar{B} & 864\cdot 3(m-3) & & \\ & (2m-2)(2m-3)W_2 & (2m-3)(m-3)W_1 & 18(4m^2-1)\bar{B} & \ddots & \\ & & (2m-3)(2m-4)W_2 & (2m-4)(m-4)W_1 & \ddots & \\ & & & (2m-4)(2m-5)W_2 & \ddots & \\ & & & & & \ddots \end{vmatrix},$$

where W_1 stands for $18\bar{B}^2-864\bar{C}$ and W_2 for $-\bar{B}^3+72\bar{B}\bar{C}-864\bar{D}$. It follows immediately that $\mathcal{D}_m(\bar{B},\bar{C},\bar{D})=\sum_{i+2j+3k=m}C_{i,j,k}\bar{B}^i\bar{C}^j\bar{D}^k$, where $C_{i,j,k}$ are integer coefficients.

PROPOSITION 2. *Provided some constants are different from zero* (*see the proof for details*) *necessary conditions for integrability in case* (ii) *assuming* $\Delta\not\equiv 0$, *are*:

(1) *If* $m\equiv 1 \pmod 6$: $\bar{B}\equiv 0$, $\bar{C}\equiv 0$ *or* $\bar{B}\equiv 0$, $\bar{D}\equiv 0$,

(2) *If* $m\equiv 2 \pmod 6$: $\bar{B}\equiv 0$, $\bar{C}\equiv 0$,

(3) *If* $m\equiv 3 \pmod 6$: $\bar{B}\equiv 0$, $\bar{D}\equiv 0$,

(4) *If* $m \equiv 4 \pmod 6$: $\bar{B} \equiv 0$, $\bar{C} \equiv 0$,

(5) *If* $m \equiv 5 \pmod 6$: $\bar{B} \equiv 0$, $\bar{C} \equiv 0$ *or* $\bar{B} \equiv 0$, $\bar{D} \equiv 0$,

the case $m \equiv 0 \pmod 6$ *being always non integrable. In the list above the following exceptions occur*:

(a) *If* $m = 1$ *the condition is only* $\bar{B} \equiv 0$,

(b) *If* $m = 2$ *it is* $\bar{b}_2 = \bar{c}_2 = 0$, $\bar{c}_1 = -3\bar{b}_1^2/256$,

(c) *If* $m = 3$ *we need* $\bar{b}_2 = 0$, $\bar{d}_2 = 65\bar{b}_1\bar{c}_2/192$, $\bar{d}_1 = 65\bar{b}_1\bar{c}_1/192 - 2881\bar{b}_1^3/1769472$.

Proof. For given values of m, $\bar{b}_1$, $\bar{b}_2$, $\bar{c}_1$, $\bar{c}_2$, $\bar{d}_1$, and $\bar{d}_2$ we shall consider $\mathscr{D}_m(\bar{B}, \bar{C}, \bar{D})$ as a function of h. It must be identically zero. This will impose conditions on the coefficients above.

If $m = 1$ then $\mathscr{D}_m = \bar{B}$ and hence $\bar{b}_1 = \bar{b}_2 = 0$.

If $m = 2$ then $\mathscr{D}_m = 8748(3\bar{B}^2 + 256\bar{C})$. The term in h^2 contains $\bar{b}_2^2$ and non-null factors. Hence $\bar{b}_2 = 0$. Then the term in h contains $\bar{c}_2$ and non-null factors. Also $\bar{c}_2$ must be zero. Finally, the independent term $3\bar{b}_1^2 + 256\bar{c}_1$ must be zero and this case is ended.

If $m = 3$ then $\mathscr{D}_m = C_{3,0,0}\bar{B}^3 + C_{1,1,0}\bar{B}\bar{C} + C_{0,0,3}\bar{D}$, where $C_{3,0,0} = -84009960$, $C_{1,1,0} = 17468006400$, $C_{0,0,1} = -51597803520$. The term in h^3 is $C_{3,0,0}\bar{b}_2^3$ and, hence, $\bar{b}_2 = 0$. Then the terms in h^1, h^0 give the other two conditions.

Now we proceed to the general cases according to the class of m modulo 6.

If $m \equiv 0 \pmod 6$ the highest power of h appears in $C_{m,0,0}\bar{B}^m$. Assuming $C_{m,0,0} \neq 0$ we should have $\bar{b}_2 = 0$. We note here that this is a general fact, independent of the value of m, provided $C_{m,0,0} \neq 0$. Then the highest power of h appears in $C_{0,m/2,0}\bar{C}^{m/2}$. Again we must have $\bar{c}_2 = 0$ if $C_{0,m/2,0} \neq 0$. But then the highest power of h appears in $C_{0,0,m/3}\bar{D}^{m/3}$, and if $C_{0,0,m/3} \neq 0$ we must have $\bar{d}_2 = 0$, but as $\bar{b}_2^2 + \bar{c}_2^2 + \bar{d}_2^2 > 0$ the system is non-integrable.

If $m \equiv 2 \pmod 6$ and $C_{m,0,0} \neq 0$, $C_{0,m/2,0} \neq 0$ we have $\bar{b}_2 = \bar{c}_2 = 0$. Now the dominant terms in h come from $\bar{B}^2\bar{D}^{(m-2)/3}$ and $\bar{C}\bar{D}^{(m-2)/3}$. As $\bar{d}_2 \neq 0$ in that case we must have $C_{2,0,(m-2)/3}\bar{b}_1^2 + C_{0,1,(m-2)/3}\bar{c}_1 = 0$. The next highest power of h appears in the $\bar{B}^5\bar{D}^{(m-5)/3}$, $\bar{B}^3\bar{C}\bar{D}^{(m-5)/3}$ and $\bar{B}\bar{C}^2\bar{D}^{(m-5)/3}$ terms. As a factor of $(\bar{d}_2 h)^{(m-5)/3}$ we have

$$C_{5,0,(m-5)/3}\bar{b}_1^5 + C_{3,1,(m-5)/3}\bar{b}_1^3\bar{c}_1 + C_{1,2,(m-5)/3}\bar{b}_1\bar{c}_1^2, \tag{19}$$

which must be zero. If $C_{0,1,(m-2)/3} \neq 0$ we obtain $\bar{c}_1$ in terms of $\bar{b}_1$ and, inserting in (19) we get

$$\bar{b}_1^5\left[C_{5,0,(m-5)/3} - C_{3,1,(m-5)/3}\frac{C_{2,0,(m-2)/3}}{C_{0,1,(m-2)/3}} + C_{1,2,(m-5)/3}\left(\frac{C_{2,0,(m-2)/3}}{C_{0,1,(m-2)/3}}\right)^2\right] = 0. \tag{20}$$

If the term inside square brackets in (20) is different from zero we should have $\bar{b}_1=0$, and then $\bar{c}_1=0$. Summarizing, we should have $\bar{B}\equiv\bar{C}\equiv 0$ but $\bar{D}$ is arbitrary, ending this case.

If $m\equiv 4 \pmod 6$ proceeding as in the previous case we have $\bar{b}_2=\bar{c}_2=0$ if $C_{m,0,0}\neq 0$, $C_{0,m/2,0}\neq 0$. Then the dominant term appears in $\bar{B}\bar{D}^{(m-1)/3}$ and, as $\bar{d}_2\neq 0$, we must have $\bar{b}_1=0$ provided $C_{1,0,(m-1)/3}\neq 0$. The next dominant term appears in $\bar{C}^2\bar{D}^{(m-4)/3}$. Again if $C_{0,2,(m-4)/3}\neq 0$ we must have $\bar{c}_1=0$, ending the proof in this case. Hence $\bar{B}\equiv 0$, $\bar{C}\equiv 0$, $\bar{D}$ being arbitrary.

We proceed to the cases with m odd. As we shall see, a part of the proof is common for the three cases. We start with the non-common part. We assume $C_{m,0,0}\neq 0$ and hence $\bar{b}_2=0$ in all cases.

If $m\equiv 1 \pmod 6$, $m>1$, the dominant terms appear in $\bar{B}\bar{C}^{(m-1)/2}$ and $\bar{C}^{(m-3)/2}\bar{D}$, and the coefficient of $h^{(m-1)/2}$ is

$$\bar{c}_2^{(m-3)/2}(C_{1,(m-1)/2,0}\bar{b}_1\bar{c}_2+C_{0,(m-3)/2,1}\bar{d}_2). \tag{21}$$

If $\bar{c}_2=0$ then the dominant term is $\bar{B}\bar{D}^{(m-1)/3}$ and, as $\bar{d}_2\neq 0$, we must have $\bar{b}_1=0$ provided $C_{1,0,(m-1)/3}\neq 0$. But then, if $C_{0,2,(m-4)/3}\neq 0$ the dominant power of h appears in $\bar{C}^2\bar{D}^{(m-4)/3}$ and we must have $\bar{c}_1=0$. Hence one possibility is $\bar{B}\equiv 0$, $\bar{C}\equiv 0$.

If $\bar{c}_2\neq 0$ the second factor in (21) must be zero. Assume $\bar{b}_1=0$ and $C_{0,(m-3)/2,1}\neq 0$. Then we must have $\bar{d}_2=0$. The current dominant term is now $C_{0,(m-3)/2,1}\,\bar{c}_2^{(m-3)/2}d_1h^{(m-3)/2}$, and we must have $\bar{d}_1=0$. Therefore another possibility is $\bar{B}\equiv 0$, $\bar{D}\equiv 0$.

It remains to discuss the case $\bar{c}_2\neq 0$, $C_{1,(m-1)/2,0}\bar{b}_1\bar{c}_2+C_{0,(m-3)/2,1}\bar{d}_2=0$, $\bar{b}_1\neq 0$. We postpone it for a joint discussion with the other m odd cases.

If $m\equiv 3 \pmod 6$ proceeding as before $\bar{b}_2=0$ and either $\bar{c}_2=0$ or $C_{1,(m-1)/2,0}\bar{b}_1\bar{c}_2+C_{0,(m-3)/2,1}\bar{d}_2=0$. If we assume $\bar{c}_2=0$ then the dominant term is $C_{0,0,m/3}\,\bar{d}_2^{m/3}h^{m/3}$, provided $C_{0,0,m/3}\neq 0$. But as $\bar{b}_2^2+\bar{c}_2^2+\bar{d}_2^2\neq 0$ this case must be discarded. Hence it is the second term which must be zero, and proceeding as in the $m\equiv 1 \pmod 6$ case, if $\bar{b}_1=0$ and $C_{0,(m-3)/2,1}\neq 0$ we must have $\bar{d}_2=0$, $\bar{d}_1=0$; i.e. we have $\bar{B}\equiv 0$, $\bar{D}\equiv 0$.

If $m\equiv 5 \pmod 6$ we must have $\bar{b}_2=0$ and either $\bar{c}_2=0$ or $C_{1,(m-1)/2,0}\bar{b}_1\bar{c}_2+C_{0,(m-3)/2,1}\bar{d}_2=0$. If we assume $\bar{c}_2=0$, the dominant power of h has the coefficient

$$(C_{2,0,(m-2)/3}\bar{b}_1^2+C_{0,1,(m-2)/3}\bar{c}_1)\,\bar{d}_1^{(m-2)/3}. \tag{22}$$

As $\bar{d}_2\neq 0$ the coefficient in (22) must be zero. The next contribution appears in $h^{(m-5)/3}$, having as coefficient

$$(C_{5,0,(m-5)/3}\bar{b}_1^5+C_{3,1,(m-5)/3}\bar{b}_1^3\bar{c}_1+C_{1,2,(m-5)/3}\bar{b}_1\bar{c}_1^2)\,\bar{d}_2^{(m-5)/3}, \tag{23}$$

and the coefficient in (23) must also be zero. Now we proceed as in the case $m \equiv 2 \pmod 6$ and, under the same assumptions on the numerical coefficients, we have $\bar{b}_1 = 0$, $\bar{c}_1 = 0$. This gives the $\bar{B} \equiv 0$, $\bar{C} \equiv 0$ case.

If $\bar{c}_2 \neq 0$ we proceed as in the $m \equiv 1 \pmod 6$ case. If $\bar{b}_1 = 0$ we also proceed as in the $m \equiv 1 \pmod 6$ case.

It remains to study the odd m cases assuming $\bar{b}_2 = 0$, $\bar{c}_2 \neq 0$, $\bar{b}_1 \neq 0$. This requires $C_{m,0,0} \neq 0$, $C_{0,(m-3)/2,1} \neq 0$. We look at the coefficients of $h^{(m-1)/2}$, $h^{(m-3)/2}$, $h^{(m-5)/2}$. Taking $\bar{C}^{(m-15)/2}$ as a factor (eventually the exponent can be negative) we should look for the coefficients of h^7, h^6 and h^5 in

$$\sum_{k=0}^{1} C_{1-k,(m-1)/2-k,k} \bar{B}^{1-k} \bar{C}^{7-k} \bar{D}^k$$
$$+ \sum_{k=0}^{3} C_{3-k,(m-3)/2-k,k} \bar{B}^{3-k} \bar{C}^{6-k} \bar{D}^k$$
$$+ \sum_{k=0}^{5} C_{5-k,(m-5)/2-k,k} \bar{B}^{5-k} \bar{C}^{5-k} \bar{D}^k. \tag{24}$$

As $\bar{c}_2 \neq 0$, $\bar{b}_1 \neq 0$, $\bar{b}_2 = 0$ we can use $\bar{E} := \bar{B}\bar{C}$ as independent variable instead of h. Then $\bar{D}$ can be written as $\mu_1 + \mu_2 \bar{E}$, where μ_1, μ_2 are suitable numerical coefficients.

Let

$$P_1(\bar{E}) = \sum_{k=0}^{1} C_{1-k,\,(m-1)/2-k,\,k}\, \bar{E}^{1-k} (\mu_1 + \mu_2 \bar{E})^k,$$
$$P_3(\bar{E}) = \sum_{k=0}^{3} C_{3-k,\,(m-3)/2-k,\,k}\, \bar{E}^{3-k} (\mu_1 + \mu_2 \bar{E})^k,$$
$$P_5(\bar{E}) = \sum_{k=0}^{5} C_{5-k,\,(m-5)/2-k,\,k}\, \bar{E}^{5-k} (\mu_1 + \mu_2 \bar{E})^k.$$

Then (24) can be written as

$$\frac{\bar{E}^6}{\bar{B}^6} P_1(\bar{E}) + \frac{\bar{E}^3}{\bar{B}^3} P_3(\bar{E}) + P_5(\bar{E}). \tag{25}$$

Looking for the terms in $\bar{E}^7$, $\bar{E}^6$, $\bar{E}^5$ in (24) and asking that they be zero we obtain

$$P_1'(0) = 0,\ P_1(0) + \frac{\bar{B}^3}{3!} P_3'''(0) = 0, \qquad \frac{1}{2!} P_3''(0) + \frac{\bar{B}^3}{5!} P_5^{V}(0) = 0,$$

or, more explicitely,

$$C_{1,(m-1)/2,0} + C_{0,(m-3)/2,1}\mu_2 = 0, \tag{26}$$

$$\mu_1 C_{0,(m-3)/2,1} + \bar{B}^3 P_3(\mu_2) = 0, \; \mu_1 P_3'(\mu_2) + \bar{B}^3 P_5(\mu_2) = 0. \tag{27}$$

From (27) we derive $C_{0,(m-3)/2,1} P_5(\mu_2) - P_3(\mu_2)\, P_3'(\mu_2) = 0$, where μ_2 is obtained from (26). If this condition is not satisfied then one should have $b_1 = 0$ and, therefore, the case $b_1 \neq 0$ must be discarded.

This ends the proof of Proposition 2 provided some numerical coefficients are non-zero. We proceed to prove this for some of them.

The coefficient $C_{m,0,0}$ is the value of $\mathscr{D}_m(\bar{B}, \bar{C}, \bar{D})$ when we set $\bar{B} = 1$, $\bar{C} = \bar{D} = 0$. Let $\Delta_{m,k}$ be the determinant obtained when in $\mathscr{D}_m$ we consider the first k rows and columns. Let $\bar{\Delta}_{m,k} = \Delta_{m,k}/18^k$. Then one has the following recurrence for $\bar{\Delta}_{m,k}$:

$$\bar{\Delta}_{m,k+1} = (4m^2 - 1)\, \bar{\Delta}_{m,k} - 48k(m-k)^2\,(2m-k)\, \bar{\Delta}_{m,k-1} \\ - 128k(m-k)(2m-k)(2m-k+1)(m-k+1)(k-1)\, \bar{\Delta}_{m,k-2}, \tag{28}$$

started with $\bar{\Delta}_{m,0} = 1$, $\bar{\Delta}_{m,1} = 1$. Of course, the desired value $\mathscr{D}_m(1, 0, 0)$ is equal to $\bar{\Delta}_{m,m}$. An elementary computation with (28) modulo 6 shows

$$\bar{\Delta}_{m,m} \equiv 1 \pmod 6 \qquad \text{if} \quad m \equiv 0, 3 \pmod 6$$

and

$$\bar{\Delta}_{m,m} \equiv 3 \pmod 6 \qquad \text{if} \quad m \equiv 1, 2, 4, 5, \pmod 6.$$

Hence $C_{m,0,0} \neq 0$.

We notice that $C_{0,m/2,0} = \mathscr{D}_m(0, 1, 0)$ if $m \equiv 0 \pmod 2$ and $C_{0,0,m/3} = \mathscr{D}_m(0, 0, 1)$ if $m \equiv 0 \pmod 3$. An easy recurrence shows

$$C_{0,m/2,0} = 864^m((m-1)!!)^2\,(2m-1)!!,$$

$$C_{0,0,m/3} = 864^m(-1)^{m/3} \binom{m}{m/3} \frac{(2m)!}{3^m},$$

and hence these coefficients are also non-zero.

The coefficients $C_{1,0,(m-1)/3}$, defined for $m \equiv 1 \pmod 3$ appear when we set $\bar{C} = 0$ in $\mathscr{D}_m(\bar{B}, \bar{C}, \bar{D})$ and we skip the terms containing $\bar{B}^2$ and $\bar{B}^3$. Furthermore, we should include only one $\bar{B}$ factor. It is immediate to check that $C_{1,0,(m-1)/3}$ is the sum of all the determinants obtained from $\mathscr{D}_m(\bar{B}, \bar{C}, \bar{D})$ when we skip the row and column of index $3k+1$, for

$k=1, \ldots, (m-1)/2$, and we set $\bar{B}=\bar{C}=0$, $\bar{D}=1$. All the terms added have the sign of $(-1)^{(m-1)/3}$. Hence $C_{1,0,(m-1)/3} \neq 0$.

In a similar way, $C_{0,(m-3)/2,1}$, defined for m odd, is seen to be negative. Indeed, its value is the sum of all the determinants of the form $\mathscr{D}_m$ when we set $\bar{B}=0$, $\bar{C}=1$ and just one $\bar{D}$ of a row of odd index equal to 1, the others being zero.

To obtain $C_{0,1,(m-2)/3}$ we set in $\mathscr{D}_m$ the variable $\bar{B}$ equal to 0. Then one of the variables $\bar{C}$, in rows of index $3k+2$, $k=0, \ldots, (m-2)/3$, is set to 1 and the other $\bar{C}$ are set to zero. $C_{0,1,(m-2)/3}$ is obtained by adding these determinants, and all of them have the sign of $(-1)^{(m-2)/3}$. Hence, it is non-zero.

Finally we proceed to show $C_{0,2,(m-4)/3} \neq 0$, this coefficient being defined for $m \equiv 1 \pmod 3$. Set $\bar{B}=0$ and consider all the possible choices of block structures for the matrix associated to $\mathscr{D}_m$, with the diagonals of the blocks contained in the diagonal of the initial matrix (that is, a block diagonal structure). We require that 2 blocks are 2×2 and the remaining ones are 3×3. In the 2×2 blocks set $\bar{C}=1$ and in the 3×3 blocks set $\bar{C}=0$, $\bar{D}=1$. Then $C_{0,2,(m-4)/3}$ is the sum of all determinants $m\times m$ obtained in this way. The sign of all of them is the one of $(-1)^{(m-4)/3}$ and this ends the proof. ∎

For the remaining coefficients to be checked that are different from zero we have not found an obvious proof. As for any specific problem they can be computed explicitely we keep this as an assumption in the statement of the proposition.

Remark 2. For convenience we list here all the assumptions made on the $C_{i,j,k}$ coefficients and not proved before. We assume $m>3$ and, of course, a coefficient $C_{i,j,k}$ is taken equal to zero if $j<0$. Let

$$\begin{aligned}\beta(m) = (-1)^{(m-4)/3} & \\ \times [\, & C_{5,0,(m-5)/3} C^2_{0,1,(m-2)/3} - C_{3,1,(m-5)/3} C_{0,1,(m-2)/3} C_{2,0,(m-2)/3} \\ & + C_{1,2,(m-5)/3} C^2_{2,0,(m-2)/3}],\end{aligned}$$

be defined for $m \equiv 2 \pmod 3$, and

$$\begin{aligned}\gamma(m) = C_{0,(m-3)/2,1} & \sum_{k=0}^{5} C_{5-k,(m-5)/2-k,k}\, \mu^k \\ & - \left(\sum_{k=0}^{3} C_{3-k,(m-3)/2-k,k}\, \mu^k \right) \left(\sum_{k=1}^{3} k C_{3-k,(m-3)/2-k,k}\, \mu^{k-1} \right),\end{aligned}$$

where $\mu = -C_{1,(m-1)/2,0} / C_{0,(m-3)/2,1}$, defined for m odd.

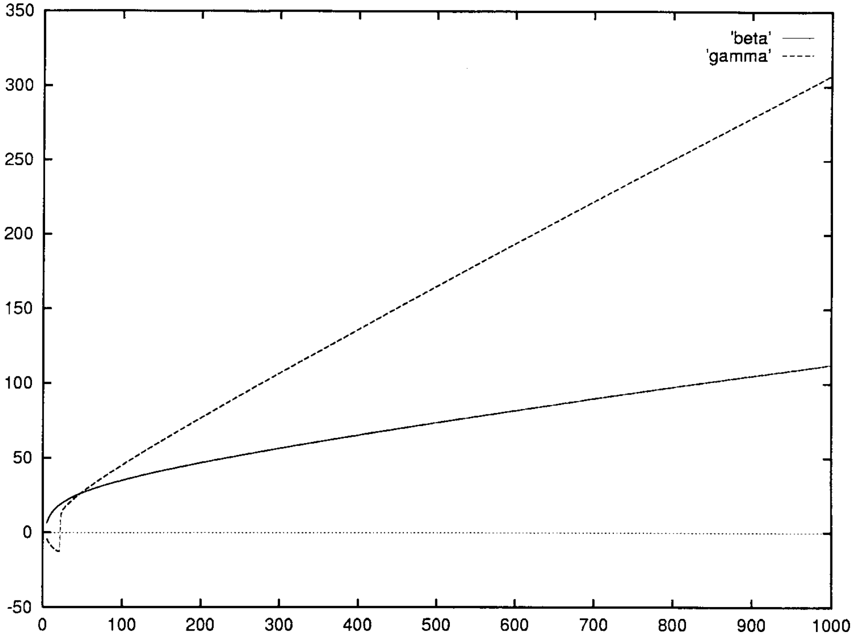

FIG. 1. The coefficients β and γ as functions of m for m between 4 and 1000. For scaling reasons we have replaced B, C, D by $2B$, $C/2$, $D/8$, respectively. For each coefficient one plots *argsh*(*coefficient*) as a function of m.

Then the assumptions of Proposition 2 are: $\beta(m)$ and $\gamma(m)$ must be non-zero when they are defined. These assumptions have been tested by the authors up to $m = 1000$ and they are satisfied in all cases. The check has been done by constructing a specific program for symbolic computation. Furthermore, beyond an eventual transient for small values of m the functions $\beta(m)$ and $\gamma(m)$ seem to increase very quickly in a quite regular way. Fig. 1 displays this behaviour.

The previous remark suggest the following.

CONJECTURE. *For all $m > 3$, when the functions β and γ are defined, they are non-zero.*

We summarize all the results of this section in the following theorem.

THEOREM 2. *Assume a classical Hamiltonian system with a potential like* (5) *has NVE of Lamé type associated to the family of solutions, Γ_h, lying on the $x_2 = 0$ plane and parametrized by the energy, h. Then, a necessary condition for integrability is that the related polynomials P_1 and P_2 (see* (10)) *satisfy $a_2 = 0$ and one of the following conditions holds*:

(1) $a_1 = 4/(n(n+1))$ *for some* $n \in \mathbb{N}$,

(2) $a_1 = 16/(4m^2 - 1)$ *for some* $m \in \mathbb{N}$. *Then, assuming the conjecture above is true, one should have* $b_2 = 0$ *and we should be in one of the following cases*:

(2.1) $m = 1$ *and* $b_1 = 0$,

(2.2) $m = 2$ *and* $c_2 = 0$, $16a_1c_1 + 3b_1^2 = 0$,

(2.3) $m = 3$ *and* $48a_1d_2 - 65b_1c_2 = 0$, $27648a_1^2d_1 - 37440a_1b_1c_1 + 2881b_1^3 = 0$,

(2.*m*) $m > 3$. *Then we should have* $b_1 = 0$ *and, furthermore, either* $c_1 = c_2 = 0$ *if* m *is congruent with* 1, 2, 4 *or* 5 *modulo* 6, *or* $d_1 = d_2 = 0$ *if* m *is odd.*

(3) $a_1 = 4/(n(n+1))$ *with* $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$, $b_2 = 0$ *and either* $c_2 = 0$, $b_1^3 - 3a_1c_1 = 0$ *or* $c_2b_1 - 3a_1d_2 = 0$, $2b_1^3 - 9a_1b_1c_1 + 27a_1^2d_1 = 0$.

Remark 3. We recall that, to be in the case of a Lamé type equation, we ask for the discriminant $\Delta(h) \not\equiv 0$. If $\Delta(h) \equiv 0$ we have several conditions on the coefficients of P_1 and P_2. If we denote by $\hat{A} = a_1 + ha_2, \ldots, \hat{D} = d_1 + hd_2$, then either $\hat{A}(h) \equiv 0$ or $R(h) \equiv 0$, where $R(h) = 27\hat{A}^2\hat{B}^2 - 18\hat{A}\hat{B}\hat{C}\hat{D} + 4\hat{A}\hat{C}^3 + 4\hat{B}^3\hat{D} - \hat{B}^2\hat{C}^2$.

5. Examples

We shall consider different examples belonging to families (A), (B) and (C) of Section 2.

(1) *Cubic Potentials.* They appear as family (A). In that case $d_2 > 0$, $\alpha = \pm\sqrt{d_2/2}\, x_1 + e$ and $P_1(\alpha) = a_1\alpha^3 + b_1\alpha^2 + c_1\alpha + d_1$, with $a_1 \neq 0$, gives immediately the potential by (10). We remark that *all* the cubic potentials of the form (5) having NVE of Lamé type associated to $x_2 = 0$ appear in this way. From Remark 3 it follows that the discriminant condition is always satisfied.

From Theorem 2 necessary conditions for integrability are that some of the following holds:

(i) $a_1 = 4/(n(n+1))$, $n \in \mathbb{N}$,

(ii) $a_1 = 16/(4m^2 - 1)$ and then: if $m = 1$, $b_1 = 0$; if $m = 2$, $16a_1c_1 + 3b_1^2 = 0$; if $m > 3$ and m is congruent to 1, 2, 4 or 5 (mod 6), $b_1 = c_1 = 0$. Other values of m give non-integrability,

(iii) $a_1 = 4/(n(n+1))$, with $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$ and then $b_1^3 = 3c_1a_1$.

In particular this includes the classical [24] and generalized [10] Hénon–Heiles potentials.

(2) *Quartic potentials.* Assume we are in the family (B) case with $P_2(\alpha) = c_2\alpha + d_2$, $c_2 \neq 0$. We recall that then $\alpha(x_1) = (c_2/8)x_1^2 + ex_1 + (2e^2 - d_2)/c_2$, e being arbitrary. Assume $P_1(\alpha) = P_2(\alpha)\, S(\alpha)$, where S is a polynomial of degree two. Then the potential V is quartic (if in the $O(x_2^3)$ terms we only include $x_2^3, x_1x_2^3, x_2^4$). Furthermore this is the only way to obtain potentials of the form (5) which are quartic. Let $S(\alpha) = s_2\alpha^2 + s_1\alpha + s_0$ and $\alpha(x_1) = \alpha_2 x_1^2 + \alpha_1 x_1 + \alpha_0$, by relabelling the coefficients. We note that $s_2, s_1, s_0, \alpha_2, \alpha_1, \alpha_0$ are arbitrary provided $s_2 \neq 0, \alpha_2 \neq 0$. This, together with the arbitrariness of the coefficients of $x_2^3, x_1x_2^3$ and x_2^4, is all the freedom available to have a quartic potential of the form (5) with NVE of Lamé type. Notice that *not* all the quartic potentials of the form (5) appear in this way. Only a codimension two subfamily. The coefficients of $P_1(\alpha)$ are $a_1 = c_2s_2$, $b_1 = c_2s_1 + d_2s_2$, $c_1 = c_2s_0 + d_2s_1$, $d_1 = d_2s_0$. From Remark 3 it follows that the discriminant condition is always satisfied.

From Theorem 2 necessary conditions for integrability are that some of the following holds:

(i) $c_2s_2 = 4/(n(n+1))$, $n \in \mathbb{N}$,

(ii) $c_2s_2 = 16/3$ and $c_2s_1 + d_2s_2 = 0$, or $c_2s_2 = 16/35$ and either $d_2 = s_1 = 0$, or $d_2s_2/65 = -c_2s_1/17 = \sqrt{52/20167}$, or, finally $c_2s_2 = 16/(4m^2-1)$, $m > 3$, m odd, and $d_2 = s_1 = 0$.

(iii) $c_2s_2 = 4/(n(n+1))$, $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$ and $c_2s_1 = 2d_2s_2$.

(3) *Rational potentials.* Again in the case of family (B) and with P_2 and α as before, assume that $P_1(\alpha) = a_1\alpha^3 + b_1\alpha^2 + c_1\alpha + d_1$ can not be divided by $P_2(\alpha)$. Then $\varphi(x_1)$ is a rational function, quotient of a polynomial of degree 6 by one of degree 2. The subfamily of rational functions of this type which can be obtained has codimension 3. Furthermore, when the rational function is given, the terms containing x_2^2 are fixed (except by multiplicative constants).

Integrability conditions are immediate from Theorem 2. We only remark that case (2.2) can not occur and case (2.m) can only occur with m odd.

(4) *Periodic Toda lattice with* 3 *particles and two equal masses.* Given the Hamiltonian with 3 degrees of freedom

$$H_1 = \frac{1}{2}\left(\frac{p_1^2}{m} + p_2^2 + p_3^2\right) + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3-q_1},$$

by means of the center of mass reduction can be simplified to

$$H=\tfrac{1}{2}(y_1^2+y_2^2)+2e^{2x_1}\cosh(2\sqrt{3\mu}\,x_2)+e^{-4x_1},$$

where μ is defined by $m=2/(3\mu-1)$. This potential is a particular case of (C.1) with

$$\alpha=-24\mu\,e^{2x_1},\quad e=-24\mu,\quad c_2=d_2=0,$$

and

$$P_2(\alpha)=8\alpha^2,\quad P_1(\alpha)=\frac{2}{3\mu}\alpha^3-8(24\mu)^2.$$

Hence $n(n+1)=4/a_1=6\mu$. If $n\notin\mathbb{N}$ the system is non integrable because $b_2\neq 0$.

(5) *Potential on a Cylinder Coinciding Locally with Hénon–Heiles.* Consider $x_1\in(-\pi/2,\pi/2)$, $x_2\in S^1$ and the potential

$$V=\tfrac{1}{2}D\sin x_1\ tg^2x_1+\tfrac{1}{2}tg^2x_1+\tfrac{1}{2}(C\sin x_1+1)\sin^2x_2.$$

It coincides with the Hénon–Heiles potential around $(0,0)$ up to third order. It is of the type (C.3.2) with $a_1=2D/(3C)$, $b_2=-2$. Hence, if $C/D\neq n(n+1)/6$, $n\in\mathbb{N}$, it is non-integrable.

6. The Homogeneous Hénon–Heiles Potential

The Hamiltonian in this case is

$$H=\frac{1}{2}(y_1^2+y_2^2)+\frac{e}{3}x_1^3+x_1x_2^2. \tag{29}$$

We can derive the integrability conditions from (1) in Section 5, but we shall proceed directly. The parameter e is assumed to be complex. We notice, first, that by a suitable scaling the potential can be also taken $\frac{1}{3}x_1^3+(1/e)x_1x_2^2$. So, the case $e=\infty$ is integrable. Furthermore, a rotation in the configuration space (eventually, a complex rotation) converts (29) into

$$H=\frac{1}{2}(\eta_1^2+\eta_2^2)+\frac{2}{3(e-1)}\xi_1^3+\xi_1\xi_2^2+\frac{\sqrt{2-e}}{3}\frac{e+1}{e-1}\xi_2^3, \tag{30}$$

if $e\neq 1$. A suitable scaling has also been used. If $e=1$, it reduces to $\frac{1}{2}(\eta_1^2+\eta_2^2)+\frac{1}{3}(\xi_1^3+\xi_2^3)$, which is clearly separable. Skipping the $O(\xi_2^3)$ terms, (30) is as (29) with $\hat{e}=2/(e-1)$.

In [22] Yoshida proves the following result:

PROPOSITION 3. *The Hamiltonian* (29) *is non-integrable for* $e \in (-\infty, 1) \cup (\bigcup_{j=1}^{\infty} (1+j+3\binom{j}{2}, 1-j+3\binom{j+1}{2}))$.

However, the complement of this set of values of e contains infinitely many intervals, with increasing lengths.

A use of the Painlevé property (see [18]) suggests that the system (29) is integrable only for $e=1$, 6, 16 and some difficulties appear in the case $e=2$ (see [8]).

Beyond the separable case $e=1$, the cases $e=6$, $e=16$ are known to be integrable. We display the first integrals independent of the Hamiltonian [18]:

(i) For $e=6$: $F=4x_1^2x_2^2+x_2^4-4x_1y_2^2+4x_2y_1y_2$,

(ii) For $e=16$: $F=y_2^4+4x_1x_2^2y_2^2-\frac{4}{3}x_2^3y_1y_2-\frac{4}{3}x_1^2x_2^4-\frac{2}{9}x_2^6$.

The first goal of this section is to prove the following non-integrability result.

PROPOSITION 4. *The Hamiltonian* (29) *is non-integrable for* $e \in \mathbb{C} \setminus \{1, 2, 6, 16\}$.

Proof. Letting aside the case $e=2$, we derive the following conditions for the coefficient n (see Introduction): $n(n+1)=12/e$ and, denoting by $\hat{n}$ the coefficient associated to $\hat{e}$: $\hat{n}(\hat{n}+1)=12/\hat{e}$. From the relation between e and $\hat{e}$ we have

$$6\left(\frac{12}{n(n+1)}-1\right)=\hat{n}(\hat{n}+1). \tag{31}$$

To be in one of the cases (i), (ii) or (iii) of Section 3, both n and $\hat{n}$ must be rationals with denominator 1, 2, 4, 6 or 10. If $n \neq 0$ (the case $n=0$ corresponding to $e=\infty$) from (31) we have

$$|\hat{n}(\hat{n}+1)| \leqslant 6\left(\frac{12}{(1/10)\cdot(9/10)}+1\right)=806.$$

Therefore $|\hat{n}|$ has an upper bound and it remains to examine a finite number of cases. A direct check shows that the only possible solutions of (31), with the required conditions, are:

(a) $n=3$, $\hat{n}=0$, corresponding to $e=1$,

(b) $n=2$, $\hat{n}=2$, corresponding to $e=2$ (notice that in this case $\hat{e}$ is also equal to 2),

(c) $n=1$, $\hat{n}=5$, corresponding to $e=6$,

(d) $n=\frac{1}{2}$, $\hat{n}=9$, corresponding to $e=16$.

This ends the proof. ∎

Now we discuss the dynamical meaning of the non-integrability. For $e<0$ it is easy to show that there are 3 simple periodic orbits, all of them touching the zero velocity curve (*zvc*) in two points. One of them is symmetrical: one can take $x_2=y_1=0$ as initial conditions. Due to the homogeneity it is enough to consider the level of energy $h=1$. Figure 2 displays the *zvc* for $e=-2$ and also the 3 simple periodic orbits (γ_1, symmetrical and γ_2, γ_3, symmetric the one of the other) projected on the (x_1, x_2) plane. These orbits are hyperbolic. For $e\nearrow 0$ the eigenvalue of largest modulus of γ_1 tends to 1 and the ones of γ_2, γ_3 to ∞. For $e\searrow -\infty$ the one of γ_1 tends to ∞ and those of γ_2, γ_3 to 1.

Figure 3 shows the intersection of γ_1 and their unstable and stable manifolds with the Poincaré section $x_2=0$. The boundary of the Poincaré section is $y_2=0$ and, hence $y_1^2/2+(e/3)x_1^3=1$, also shown in Fig. 3. The invariant manifolds intersect transversally at an homoclinic point and this implies unpredictable dynamics. Similar patterns appear for any $e<0$ (but they are difficult to see for $|e|$ small, for instance).

The case $e\geqslant 0$ is more subtle. One has $\ddot{x}_1=-ex_1^2-x_2^2$ and, as the x_1 acceleration is always $\ddot{x}_1\leqslant 0$, there is no possible recurrence in the real phase space. We can look for it in the complex phase space. Let 1_α denote a complex number of modulus 1 and argument α.

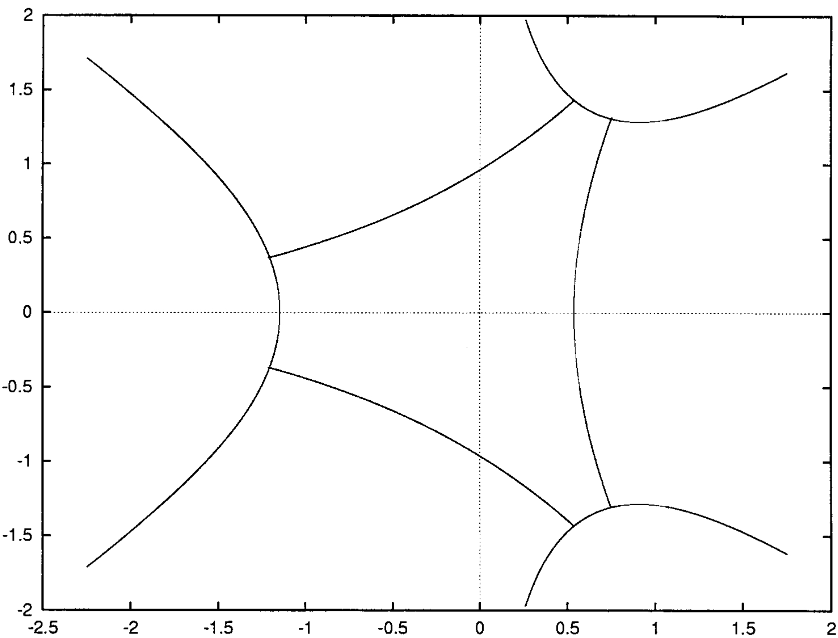

FIG. 2. Zero velocity curve and (x_1, x_2) projections of the 3 simple periodic orbits, for $e=-2$.

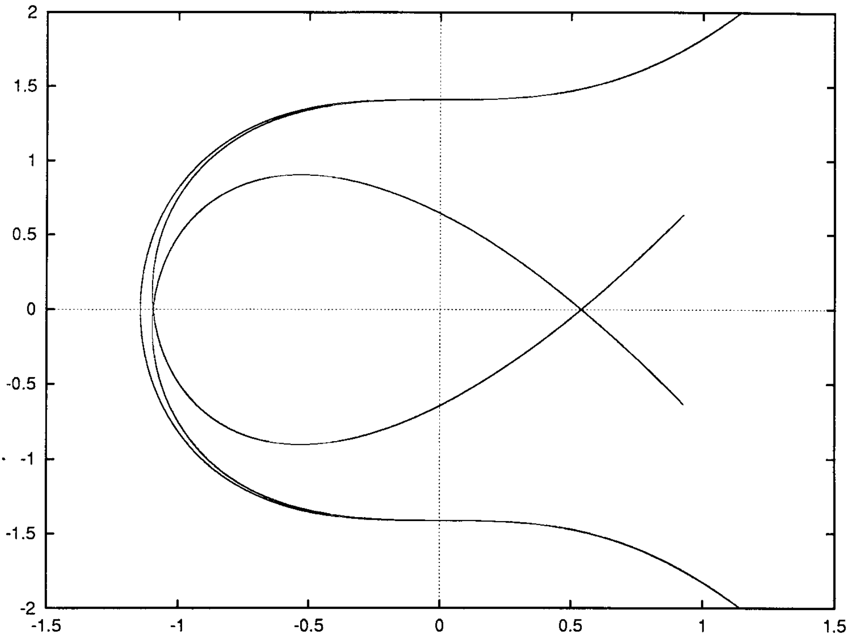

FIG. 3. The boundary of the Poincaré section through $x_2=0$ on the (x_1, y_1) coordinates and the sections of the invariant manifolds of γ_1 for $e=-2$ and $h=1$.

The changes

$$x_1 = 1_{\pi/3}\xi_1,\ x_2 = 1_{-\pi/6}\xi_2,\ y_1 = 1_{-\pi/2}\eta_1,\ y_2 = \eta_2,\ t = 1_{-\pi/6}s$$

lead to the Hamiltonian (using s as new time)

$$H = \frac{1}{2}\left(-\eta_1^2 + \eta_2^2\right) + \frac{e}{3}\left(-\xi_1^3\right) + \xi_1\xi_2^2, \tag{32}$$

which is real for real variables and on the same level of energy. For this one the acceleration changes sign. Take, for instance, the value $e=3/2$. Figure 4 shows a Poincaré section of (32) on $h=1$, $\xi_2=0$. One can see a typical pattern with invariant curves, islands and chaotic regions. Symmetric periodic orbits appear for $\xi_1^e = 1.346146...$ (elliptical) and $\xi_1^h = 1.766010...$ (hyperbolic). In Fig. 4 initial points are taken with $\xi_1 = \xi_1^e + \delta, \eta_1 = 0$, where $\delta = 0.335(0.01)0.405$. In fact the last of these points is close to the hyperbolic one and escapes after a few thousands of iterates of the Poincaré map. A similar behaviour is observed for other values of e. However, for e near to 1.51602386 the elliptic and hyperbolic simple symmetric periodic orbits coincide in a parabolic orbit and they can not be continuated to

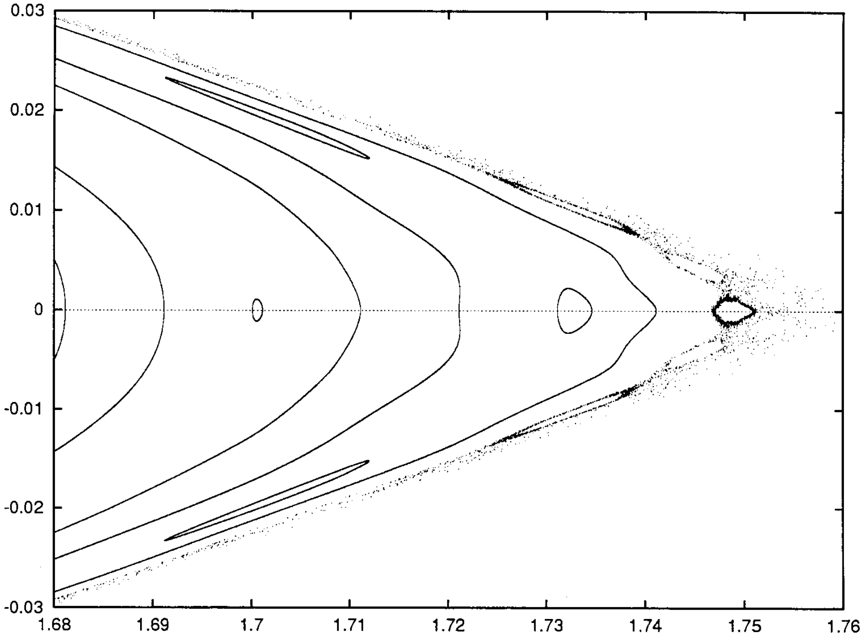

FIG. 4. A part of the phase portrait of the Poincaré map through $\xi_2 = 0$ on the (ξ_1, η_1) coordinates for $e = 1.5$ and $h = 1$.

larger values with real (ξ, η) variables (of course, there are not periodic orbits with real (x, y) variables). We are interested in the case $e = 2$.

It is an easy matter to move e to the complex plane, to follow a path and obtain the corresponding symmetric ($x_2 = 0$ and $y_1 = 0$ initially) complex periodic orbits (of real dimension 1), with complex period. We remark that the passage from e to the initial value x_1^0 is an interesting Riemann surface (we keep $h = 1$), having branches when the eigenvalues are equal to 1.

For $e = 2$ the initial value $x_1^0 \simeq 0.01247621 + i\,1.08807831$, leads to a periodic orbit with period $T \simeq 5.25449302 - i\,2.60364041$ and dominant eigenvalue $\lambda \simeq -3.39790418 + i\,27.26367123$. We keep the Poincaré section $x_2 = 0$ (as complex). The unstable and stable invariant manifolds have been generated, numerically, from a fundamental domain (in the Poincaré section) diffeomorphic to a real 2D annulus. The intersections of these manifolds with this section on $h = 1$ are complex symmetric curves (of real dimension 2). If $x_1 = g_u(y_1)$, $x_1 = g_s(y_1)$ describe, locally, the unstable and stable manifolds respectively, one has $g_s = -g_u$. An homoclinic point has been found near $x_1 = 1.15441741 + i\,0.14795498$, $y_1 = 0$. At this point one has $dx_1/dy_1 \simeq -0.0514 + 0.1042\,i$. Hence the manifolds intersect transversally and unpredictable motion occurs. In particular this prevents the integrability for $e = 2$.

The preceeding discussion leads to the following natural questions:

QUESTION 1. *Is it true that if a system is non integrable, in the sense used in this paper, unpredictability occurs in some part (eventually complex) of the phase space?*

QUESTION 2. *Is the Hénon–Heiles homogeneous potential integrable for all $e > 0$ if we restrict to the real phase space?*

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